On α^+-Stable König-Egervary Graphs

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Abstract

The stability number of a graph G, denoted by $\alpha(G)$, is the cardinality of a stable set of maximum size in G. If its stability number remains the same upon the addition of any edge, then G is called α^+-stable. G is a König-Egervary graph if its order equals $\alpha(G)+\mu(G)$, where $\mu(G)$ is the cardinality of a maximum matching in G. In this paper we characterize α^+-stable König-Egervary graphs, generalizing some previously known results on bipartite graphs and trees. Namely, we prove that a König-Egervary graph $G=(V,E)$ is α^+-stable if and only if either $|\cap\{V-S : S \in \Omega(G)\}| = 0$, or $|\cap\{V-S : S \in \Omega(G)\}| = 1$, and G has a perfect matching (where $\Omega(G)$ denotes the family of all maximum stable sets of G). Using this characterization we obtain several new findings on general König-Egervary graphs, for example, the equality $|\cap\{S : S \in \Omega(G)\}| = |\cap\{V-S : S \in \Omega(G)\}|$ is a necessary and sufficient condition for a König-Egervary graph G to have a perfect matching.

1 Introduction

Throughout this paper $G=(V,E)$ is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set $V=V(G)$ and edge set $E=E(G)$. If $X \subset V$, then $G[X]$ is the subgraph of G spanned by X. By $G-W$ we mean the subgraph $G[V-W]$, if $W \subset V(G)$. By $G-F$ we denote the partial subgraph of G obtained by deleting the edges of F, for $F \subset E(G)$, and we use $G-e$, if $W=\{e\}$. If $A,B \subset V$ and $A \cap B = \emptyset$, then (A,B) stands for the set $\{e=ab : a \in A, b \in B, e \in E\}$.

A *stable set* in G is a set $A \subseteq V$ of pairwise non-adjacent vertices. A stable set of maximum size will be referred as to a *maximum stable set* of G and its cardinality $\alpha(G)$ is the *stability number* of G. Let $\Omega(G)$ stand for the set $\{S : S$ *is a maximum stable set of* $G\}$.

A matching (i.e., a set of non-incident edges of G) of maximum cardinality $\mu(G)$ is a *maximum matching*, and a *perfect matching* is one covering all the vertices of G. If $|V(G)| - 2|M| = 1$, then M is called *near-perfect*, [22]. By C_n, K_n, P_n we denote

the chordless cycle on $n \geq 4$ vertices, the complete graph on $n \geq 1$ vertices, and respectively the chordless path on $n \geq 3$ vertices.

It is known that $\lfloor n/2 \rfloor + 1 \leq \alpha(G) + \mu(G) \leq n$ holds for any graph G with n vertices. Any complete graph K_n represents the lower bound in this inequality, while the upper bound is achieved, according to a well-known result of Koenig, [15], and Egervary, [7], by any bipartite graph. It is easy to see that there are also non-bipartite graphs having the same property, for instance, the graphs in Figure 1.

Figure 1: König-Egervary non-bipartite graphs.

If $\alpha(G) + \mu(G) = |V(G)|$, then G is called *a König-Egervary graph*. We attribute this definition to Deming [5], and Sterboul [26], but it is also possible to say that Klee [14] defined this notion implicitly before them. These graphs were studied by Korach [16], Lovasz [21], Lovasz and Plummer [22], Bourjolly and Pulleyblank [3], Pulleyblank [25], and generalized by Bourjolly, Hammer and Simeone [2], Paschos and Demange [24]. Since G is a König-Egervary graph if and only if all its connected components are König-Egervary graphs, throughout this paper we shall consider only connected König-Egervary graphs.

A graph G is α^+-*stable* if $\alpha(G + e) = \alpha(G)$ holds for any edge $e \in E(\overline{G})$, where $\overline{G}$ is the complement of G, [11]. We shall use the following characterization that Haynes et al. give for the α^+-stable graphs.

Theorem 1.1 *[13] A graph G is α^+-stable if and only if $|\cap\{S : S \in \Omega(G)\}| \leq 1$.*

Theorem 1.1 motivates us to define graph G as α_0^+-*stable* if $|\cap\{S : S \in \Omega(G)\}| = 0$, and α_1^+-*stable* if $|\cap\{S : S \in \Omega(G)\}| = 1$, [20]. Based on Theorem 1.1, Gunther et al., [11], give a description of α^+-stable trees, which we generalized to bipartite graphs in [18]. The structure of α^+-stable bipartite graphs is emphasized in [19].

In this paper we present several properties of König-Egervary graphs, which we use further to give necessary and sufficient conditions for König-Egervary graphs to be α^+-stable. We also characterize König-Egervary graphs having perfect matchings. Similar problems related to adding or deleting edges or vertices in connection with various graph parameters are treated in [1], [4], [8], [9], [23], [27].

2 König-Egervary Graphs

Using the definition of König-Egervary graphs we get:

Lemma 2.1 *(i) If G is a König-Egervary graph, then $\alpha(G) \geq |V(G)|/2 \geq \mu(G)$.*

(ii) A König-Egervary graph G has a perfect matching if and only if $\alpha(G) = \mu(G)$.

(iii) If G admits a perfect matching, then $\alpha(G) = \mu(G)$ if and only if G is a König-Egervary graph.

For $G_i, i = 1, 2$, let $G = G_1 * G_2$ be the graph with $V(G) = V(G_1) \cup V(G_2)$, and $E(G) = E(G_1) \cup E(G_2) \cup \{xy : \textit{for some } x \in V(G_1) \textit{ and } y \in V(G_2)\}$. Clearly, if H_1, H_2 are subgraphs of a graph G such that $V(G) = V(H_1) \cup V(H_2)$ and $V(H_1) \cap V(H_2) = \emptyset$, then $G = H_1 * H_2$, i.e., any graph of order at least two admits such decompositions. However, some particular cases are of special interest. For instance, if: $E(H_i) = \emptyset, i = 1, 2$, then $G = H_1 * H_2$ is bipartite; $E(H_1) = \emptyset$ and H_2 is complete, then $G = H_1 * H_2$ is a *split graph* [10].

The following proposition shows that the König-Egervary graphs are, in this sense, between these two "extreme" situations. The equivalence of the first and the third parts of this result was proposed by Klee without proof (see [14]).

Proposition 2.2 *If G is connected, then the following statements are equivalent:*

(i) G is a König-Egervary graph;

(ii) $G = H_1 * H_2$, where $V(H_1) = S \in \Omega(G)$ and $|V(H_1)| \geq \mu(G) = |V(H_2)|$;

(iii) $G = H_1 * H_2$, where $V(H_1) = S$ is a stable set in G, $|S| \geq |V(H_2)|$, and $(S, V(H_2))$ contains a matching M with $|M| = |V(H_2)|$.

Proof. $(i) \Rightarrow (ii)$ Let $S \in \Omega(G), H_1 = G[S]$ and $H_2 = G[V - S]$. Then we have $G = H_1 * H_2, \alpha(G) + \mu(G) = |V(G)| = \alpha(G) + |V(H_2)|$, and therefore $\mu(G) = |V(H_2)|$. In addition, Lemma 2.1 ensures that $|V(H_1)| \geq \mu(G)$.

$(ii) \Rightarrow (iii)$ It is clear if we take the same H_1 and H_2 as in (ii).

$(iii) \Rightarrow (i)$ First, we claim that $|M| = \mu(G)$. To see this, let assume W be an arbitrary matching in G containing some edge of H. Since S is stable, we infer that $|W| < |V(H_2)| = |M|$. Therefore, M must be a maximum matching in G. Hence we have: $\alpha(G) + \mu(G) \leq |V(G)| = |S| + |V(H)| = |S| + |M| = |S| + \mu(G)$, and because S is stable, we obtain that $|S| = \alpha(G)$ and $\alpha(G) + \mu(G) = |V(G)|$, i.e., G is a König-Egervary graph. ■

In the sequel, we shall often represent a König-Egervary graph G as $G = S * H$, where $S \in \Omega(G)$, $H = G[V - S]$, and $|V(H)| = \mu(G)$.

Lemma 2.3 *Any maximum matching of a König-Egervary graph $G = (V, E)$ is contained in each $(S, V - S)$, where $S \in \Omega(G)$, and, hence,*

$$\cup\{M : M \textit{ is a maximum matching in } G\} \subseteq \cap\{(S, V - S) : S \in \Omega(G)\}.$$

Proof. Let $S \in \Omega(G)$ and $G = S * H$. Suppose, on the contrary, that there is a maximum matching M of G and an edge $e = xy \in M \cap E(H)$. Since S is stable, we infer that $\mu(G) < |V(H)|$, a contradiction. Therefore, M must be contained in $(S, V - S)$. ■

Let M be a maximum matching of a graph G. To adopt Edmonds's terminology, [6], we recall the following terms for G relative to M. The edges in M are *heavy*, while those not in M are *light*. An *alternating path* from a vertex x to a vertex y is a x, y-path whose edges are alternating light and heavy. A vertex x is *exposed* relative to M if x is not the endpoint of a heavy edge. An odd cycle C with $V(C) = \{x_0, x_1, ..., x_{2k}\}$ and $E(C) = \{x_i x_{i+1} : 0 \leq i \leq 2k - 1\} \cup \{x_{2k}, x_0\}$, such that $x_1x_2, x_3x_4, ..., x_{2k-1}x_{2k} \in M$ is a *blossom relative to M*. The vertex x_0 is the *base* of the blossom. The *stem* is

an even length alternating path joining the base of a blossom and an exposed vertex for M. The base is the only common vertex to the blossom and the stem. A *flower* is a blossom and its stem. A *posy* or a *blossom pair* (cf. [5]) consists of two (not necessarily disjoint) blossoms joined by an odd length alternating path whose first and last edges belong to M. The endpoints of the path are exactly the bases of the two blossoms. The following result of Sterboul, [26], characterizes König-Egervary graphs in terms of forbidden configurations.

Theorem 2.4 *For a graph G, the following properties are equivalent:*
(i) G is a König-Egervary graph;
(ii) there exist no flower and no posy relative to some maximum matching M;
(iii) there exist no flower and no posy relative to any maximum matching M.

If a König-Egervary graph G is blossom-free relative to a maximum matching M, then G is not necessarily blossom-free with respect to any of its maximum matchings. For instance, the graph G in Figure 2 contains a unique C_5, which is a blossom relative to the maximum matching $M_1 = \{d, e, g\}$, and is not a blossom relative to $M_2 = \{a, c, g\}$.

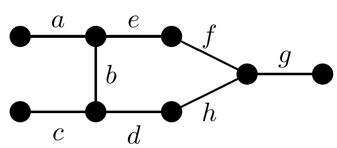

Figure 2: Graph G is not blossom-free.

Lemma 2.5 *If M is a maximum matching and S is a stable set of a König-Egervary graph G, then $S \in \Omega(G)$ if and only if S contains all exposed vertices relative to M and one endpoint of each edge in M.*

Proof. According to Proposition 2.2, $G = S * H$, where $S \in \Omega(G)$ and $H = G[V - S]$ has $\mu(G) = |V(H)|$. By Lemma 2.3, $M \subset (S, V - S)$, and therefore the assertion on S is true.

Conversely, since S is stable and $|S| = |M| + |V(G)| - 2|M| = |V(G)| - |M| = |V(G)| - \mu(G)$, we get that $S \in \Omega(G)$. ■

Theorem 2.6 *Let G be a König-Egervary graph of order at least* 2*. Then G satisfies* $|\cap\{V - S : S \in \Omega(G)\}| = 0$ *if and only if it has a perfect matching and is blossom-free.*

Proof. Suppose that G has no perfect matching. Let $S \in \Omega(G), G = S * H$, and M be a maximum matching in G. Lemma 2.1 implies that $\alpha(G) > \mu(G)$, and hence, G has at least one exposed vertex v with respect to M. Then $v \in S$, and any $w \in N(v)$ is not contained in S. Since the choice of S is arbitrary, we conclude that $w \in V - S$, for any $S \in \Omega(G)$. Hence, $|\cap\{V - S : S \in \Omega(G)\}| > 0$, in contradiction with the premises on G. Thus, G must have a perfect matching. To prove that G is blossom-free, it is sufficient to show that if G is a König-Egervary graph, then $\{x : x$ *is a base of a*

blossom in $G\} \subseteq \cap\{V - S : S \in \Omega(G)\}$, i.e., for any $S \in \Omega(G)$, no base of a blossom in G belongs to S. Let C be a blossom in G, with $V(C) = \{x_0, x_1, ..., x_{2k}\}$, relative to a perfect matching M, and x_0 be its base. Then $x_1x_2, x_3x_4, ..., x_{2k-1}x_{2k} \in M$, and according to Lemma 2.5, S contains one of the vertices x_1 or x_{2k}. If $x_1,x_{2k} \notin S$, then necessarily $x_2, x_{2k-1} \in S$ and this is not possible, since the node distance on C between x_2 and x_{2k-1} is an even number. Hence, $x_0 \notin S$.

Conversely, Let M be a perfect matching of $G = S * H$, and $b \in V(H)$, where $S \in \Omega(G)$, $H = G[V - S]$, and $|V(H)| = \mu(G)$. We emphasize a maximum stable set of G that contains b. Let denote:

$$A_1 = N(b) \cap S, B_1 = \{b : ab \in M, a \in A_1\},$$

$$A_2 = N(B_1) \cap S - A_1, B_2 = \{b : ab \in M, a \in A_2\},$$

$$A_3 = N(B_2) \cap S - A_1 \cup A_2, B_3 = \{b : ab \in M, a \in A_3\}, ...,$$

$$A_p = N(B_{p-1}) \cap S - A_1 \cup ... \cup A_{p-1}, B_p = \{b : ab \in M, a \in A_p\},$$

and $b \in B = B_1 \cup ... \cup B_p, A = A_1 \cup ... \cup A_p$ be such that $(B, S - A) = \emptyset$. Any edge joining two vertices in B would close a blossom with respect to M, which contradicts the fact that G is blossom free. Therefore, B is stable. The set $B \cup (S - A)$ is also stable, because $(B, S - A) = \emptyset$. Moreover, $|B| = |A|$ implies that $B \cup (S - A)$ is a maximum stable set of G.

Thus, every $b \in V(H) = V - S$ belongs to a maximum stable set. Since S is also a maximum stable set, we conclude that any vertex of G belongs to some maximum stable set of G. Clearly, this is equivalent to $|\cap\{V - S : S \in \Omega(G)\}| = 0$. ■

It is worth observing that having a perfect matching is not sufficient for achieving $|\cap\{V - S : S \in \Omega(G)\}| = 0$. For instance, $K_4 - e$ is a König-Egervary graph with perfect matchings, but $|\cap\{V - S : S \in \Omega(G)\}| = 2$. Being blossom free is also not enough for $|\cap\{V - S : S \in \Omega(G)\}| = 0$. For instance, trees without a perfect matching are examples of blossom free graphs such that $|\cap\{V - S : S \in \Omega(G)\}| \neq 0$.

3 α^+-Stable König-Egervary Graphs

Lemma 3.1 *Any* α^+*-stable König-Egervary graph has a near-perfect matching or a perfect matching.*

Proof. Suppose that graph G has neither a near-perfect matching nor a perfect matching. Let M be a maximum matching of G. Since $|V(G)| - 2|M| \geq 2$, there exist two unmatched vertices of G, say x, y. Hence, $e = xy \in E(\overline{G})$, because otherwise $M \cup \{e\}$ is a matching larger than a maximum matching of G. We claim that x, y are contained in all maximum stable sets of G. To see this, let $S \in \Omega(G)$ and $H = G[V - S]$. Then $G = S * H$, where $\mu(G) = |V(H)| = |M|$. By Lemma 2.3, we have that $M \subseteq (S, V - S)$. Hence, $x, y \in S$, because these vertices are unmatched and non-adjacent. Since S was an arbitrary maximum stable set of G, we infer that $x, y \in \cap\{S : S \in \Omega(G)\}$. By Theorem 1.1, it contradicts the fact that G is α^+-stable. Consequently, G must have a near-perfect matching or a perfect matching. ■

Theorem 3.2 *A König-Egervary graph G is α^+-stable if and only if it has a perfect matching and $|\cap\{V-S : S \in \Omega(G)\}| \leq 1$.*

Proof. Let G be α^+-stable, and $S \in \Omega(G)$. Suppose, on the contrary, that G has no perfect matching, i.e., by Lemma 2.1, $\alpha(G) > \mu(G)$. Lemma 3.1 implies that G has a near-perfect matching M, which is contained, according to Lemma 2.3, in $(S, V-S)$. Hence, we get that $\alpha(G) = |S| = \mu(G) + 1 = |V-S| + 1 = |M| + 1$, and there are $x, y \in S$ and $z \in V-S$ such that $xz \in E(G) - M$ and $yz \in M$. We claim that x, y belong also to any other maximum stable set W of G, since otherwise if:

(a) $z \in W$, then $x, y \notin W$, and hence $|W| < \alpha(G)$, a contradiction;

(b) only $x \in W$ or only $y \in W$, then $z \notin W$, and again the contradiction $|W| < \alpha(G)$, because all vertices of $S - \{x, y\}$ are respectively matched, by M-$\{yz\}$, with vertices in $V - S - \{z\}$.

Thus, we get that $x, y \in \cap\{S : S \in \Omega(G)\}$, and according to Theorem 1.1, this contradicts the fact that G is α^+-stable. Therefore, G has a perfect matching, say M. Then, for any edge $e = xy \in M$, we have that $x \in \cap\{S : S \in \Omega(G)\}$ if and only if $y \in \cap\{V-S : S \in \Omega(G)\}$. Consequently, we obtain that $|\cap\{S : S \in \Omega(G)\}| = |\cap\{V-S : S \in \Omega(G)\}|$, and Theorem 1.1 implies $|\cap\{V-S : S \in \Omega(G)\}| \leq 1$.

Conversely, suppose G has a perfect matching and $|\cap\{V-S : S \in \Omega(G)\}| \leq 1$. As we saw in the previous paragraph, the existence of a perfect matching in G results in $|\cap\{S : S \in \Omega(G)\}| = |\cap\{V-S : S \in \Omega(G)\}|$. Since $|\cap\{V-S : S \in \Omega(G)\}| \leq 1$, Theorem 1.1 ensures that G is α^+-stable. ■

It is worth mentioning that there are König-Egervary graphs with perfect matchings, which are not α^+-stable; e.g., the graph $K_4 - e$. However, for bipartite graphs, this condition is also sufficient (see Corollary 4.1).

Propositions 3.3 and 3.4 show that any α^+-stable König-Egervary graph G with $|\cap\{V-S : S \in \Omega(G)\}| = 1$ may be decomposed into two α^+-stable König-Egervary graphs G_1, G_2 with $|\cap\{V-S : S \in \Omega(G_1)\}| = 0$, and $|\cap\{V-S : S \in \Omega(G_2)\}| = 0$.

Proposition 3.3 *If G is a König-Egervary graph with $|\cap\{V-S : S \in \Omega(G)\}| = 1$ and $\alpha(G) = \mu(G)$, then there exists $xy \in E(G)$, such that $H = G - \{x, y\}$ is a König-Egervary graph with $|\cap\{V-S : S \in \Omega(H)\}| = 0$ and $\alpha(H) = \mu(H)$.*

Proof. Let M be a perfect matching of G, which exists by Lemma 2.1. Let also $x \in \cap\{V-S : S \in \Omega(G)\}$ and $y \in V(G)$ be such that $e = xy \in M$. Hence, it follows that $y \in \cap\{S : S \in \Omega(G)\}$, and therefore, $H = G - \{x, y\}$ is a König-Egervary graph with $\alpha(H) = \mu(H)$ and $|\cap\{V-S : S \in \Omega(H)\}| = 0$. ■

Proposition 3.4 *If G is a König-Egervary graph with $|\cap\{V-S : S \in \Omega(G)\}| = 0$ and $K_2 = \{\{x, y\}, \{xy\}\}$, then every graph $F = G + K_2$ having:*

$$V(F) = V(G) \cup \{x, y\}, E(F) \supseteq E(G) \cup \{xy\} \text{ and } \{(y, S) \neq \emptyset, \text{ for all } S \in \Omega(G)\},$$

is a König-Egervary graph with a perfect matching, and $|\cap\{V-S : S \in \Omega(F)\}| = 1$.

Proof. By Theorem 2.6, we obtain that G admits perfect matchings. Since $(y, S) \neq \emptyset$ for any $S \in \Omega(G)$, we get $\Omega(F) = \{S \cup \{x\} : S \in \Omega(G)\}$. $M \cup \{xy\}$ is a perfect matching in F, for any perfect matching M of G. Consequently, F has a perfect matching and $\cap \{V - S : S \in \Omega(F)\} = \{y\}$. According to Proposition 2.2, F is also a König-Egervary graph. ■

The next theorem presents a more specific characterization of α^+-stable König-Egervary graphs.

Theorem 3.5 *If G is a König-Egervary graph of order at least 2, then the following statements are equivalent:*

(i) G is α^+-stable;

(ii) either $|\cap\{V - S : S \in \Omega(G)\}| = 0$, or $|\cap\{V - S : S \in \Omega(G)\}| = 1$, and G has a perfect matching;

(iii) G has a perfect matching, and either there exists $xy \in E(G)$, such that $H = G - \{x, y\}$ is blossom-free and has a perfect matching, or G is blossom-free.

Proof. (iii) If $|\cap\{V - S : S \in \Omega(G)\}| = 0$, then G has a perfect matching and it is blossom-free, by Theorem 2.6. If $|\cap\{V - S : S \in \Omega(G)\}| = 1$ and G has a perfect matching, then Proposition 3.3 and Lemma 2.1, imply that there exists $xy \in E(G)$, such that $H = G - \{x, y\}$ is blossom-free and has a perfect matching.

$(iii) \Rightarrow (i)$ If G has a perfect matching and is blossom-free, then according to Theorem 2.6, we get $|\cap\{V - S : S \in \Omega(G)\}| = 0$, and further, Theorem 3.2 ensures that G is α^+-stable. If G has a perfect matching, and there exist $xy \in E(G)$, such that $H = G - \{x, y\}$ is blossom-free and has a perfect matching, then G is α^+-stable, according to Proposition 3.4. ■

The graph $K_4 - e$ shows that it is not enough to have a perfect matching in order to ensure that a König-Egervary graph is α^+-stable.

Notice also that P_3 is a König-Egervary graph, $|\cap\{V - S : S \in \Omega(P_3)\}| = 1$, but P_3 is not α^+-stable.

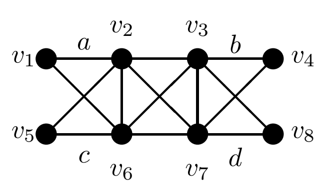

Figure 3: G is not-blossom-free, $H = G - \{v_2, v_3\}$ is blossom-free, and $\alpha(H) \neq \mu(H)$.

Observe that the graph G, in Figure 3, has blossoms with respect to the perfect matching $M = \{a, b, c, d\}$, and only for $x \in \{v_2, v_6\}$ and $y \in \{v_3, v_7\}$, the corresponding subgraph $H = G - \{x, y\}$ is connected and blossom-free, but $\alpha(H) \neq \mu(H)$. In addition, G is a König-Egervary non-α^+-stable graph, since $\alpha(G + v_1v_5) = 3 < \alpha(G)$.

4 Applications of α^+-stable König-Egervary Graphs

Combining Theorems 3.5, 1.1 and Lemma 2.1, we obtain:

Corollary 4.1 *[18] If G is a bipartite graph, then the following assertions are equivalent:*
(i) G is α^+-stable;
(ii) G possesses a perfect matching;
(iii) G has two maximum stable sets that partition its vertex set;
(iv) $|\cap\{S : S \in \Omega(G)\}| = 0$.

In other words, the bipartite graphs can be only α_0^+-stable. Nevertheless, there exist non-bipartite König-Egervary α_0^+-stable graphs (e.g., G_2 in Figure 4), and also non-bipartite König-Egervary α_1^+-stable graphs (e.g., G_1 in Figure 4).

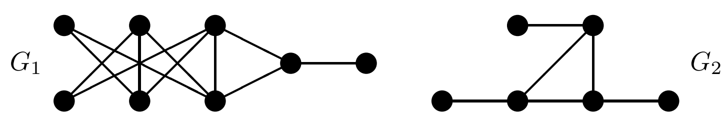

Figure 4: α^+-stable non-bipartite König-Egervary graphs.

Proposition 4.2 *Let G be an α^+-stable bipartite graph. If graph $H = G \bullet K_p$ has $V(H) = V(G) \cup V(K_p)$ and $E(H) = E(G) \cup E(K_p) \cup W$, where:*
(i) $W = \{xa, xb\}$, with ab in a perfect matching of G, for $p \leq 2$ and $x \in V(K_p)$; or
(ii) $W = \{xy\}$, for some $x \in V(K_p)$ and $y \in V(G)$, for $p \geq 3$,
then H is α^+-stable.

Proof. *(i)* If $p = 1$, we claim that $\alpha(H) = \alpha(G)$. Otherwise, there is a stable set S in H with $|S| = \alpha(H) > \alpha(G)$, and consequently $S - \{x\}$ is a maximum stable set in G that contains neither a, nor b, a contradiction, since G is α^+-stable. Hence, we have $\Omega(H) = \Omega(G)$ and clearly, $\cap\{S : S \in \Omega(H)\} = \emptyset$. If $p = 2$, then $|\cap\{S : S \in \Omega(H)\}| = |\{y\}| 1$, because G is α^+-stable and any maximum stable set of H is of the form $S \cup \{y\}$, where $S \in \Omega(G)$ and $y \in V(K_2) - \{x\}$. Therefore, by Theorem 1.1, H is α^+-stable.

(ii) In this case, $\cap\{S : S \in \Omega(H)\} = \emptyset$, because G is α^+-stable and any maximum stable set of H is of the form $S \cup \{z\}$, where $S \in \Omega(G)$ and $z \in V(K_p) - \{x\}$. According to Theorem 1.1, H is α^+-stable. ■

The graph $G = K_1 \bullet K_p$, for $p \geq 4$, $K_1 = (\{x\}, \emptyset), V(G) = \{x\} \cup V(K_p)$ and $E(G) = \{xy\} \cup E(K_p)$, where $y \in V(K_p)$, is α^+-stable, non-König-Egervary graph, and $|\cap\{S : S \in \Omega(G)\}| = |\{x\}| = 1$. Taking also into account Proposition 4.2, we obtain the following:

Corollary 4.3 *For every natural number $n \geq 5$ there exist α^+-stable non-König-Egervary graphs G_1, G_2 of order n such that*

$$|\cap\{S : S \in \Omega(G_1)\}| = 0 \textit{ and } |\cap\{S : S \in \Omega(G_2)\}| = 1.$$

Proposition 4.4 *If G is a König-Egervary graph of order $n \geq 2$, and $\alpha(G) > n/2$, then $|\cap\{S : S \in \Omega(G)\}| \geq 2$.*

Proof. If $\alpha(G) > n/2$, then G has no perfect matching, and by Theorem 3.2, G is not α^+-stable. Consequently, Theorem 1.1 implies that $|\cap\{S : S \in \Omega(G)\}| \geq 2$. ■

For general case, it has been proven that:

Proposition 4.5 *[12] If G is a graph with $\alpha(G) > |V(G)|/2$, then*

$$|\cap\{S : S \in \Omega(G)\}| \geq 1.$$

For example, in a bipartite graph $G = (A, B, E)$ such that $|A| \neq |B|$, there exists at least one vertex belonging to all maximum stable sets of G, i.e.,

$$|\cap\{S : S \in \Omega(G)\}| \geq 1.$$

Since any bipartite graph is König-Egervary, Proposition 4.4 yields the following result, which has been already done independently in [19], as a strengthening of Proposition 4.5 in the case of bipartite graphs.

Corollary 4.6 *If $G = (A, B, E)$ is bipartite and $|A| \neq |B|$, then*

$$|\cap\{S : S \in \Omega(G)\}| \geq 2.$$

A *pendant edge* is an edge incident with a *pendant vertex* (i.e., a vertex of degree one). A vertex v is *α-critical* in G if $\alpha(G - v) < \alpha(G)$.

Theorem 4.7 *For a graph G of order at least two, the following are equivalent:*
(i) G has a perfect matching M consisting of its pendant edges;
(ii) G has exactly $\alpha(G)$ pendant vertices and none of them is α-critical;
(iii) G is a König-Egervary α^+-stable graph with exactly $\alpha(G)$ pendant vertices.

Proof. *(i)* $\Rightarrow$ *(ii)* It is clear that $S = \{x : x \textit{ is a pendant vertex in } G\}$ is stable in G. If $\alpha(G) > |S| = |V(G)|/2$, then any maximum stable set W of G must contain some pair of vertices, matched by M, a contradiction, since W is stable. Hence, $|S| = \alpha(G)$ holds. In addition, if $x \in S$ and y is its single neighbor in G, then $S \cup \{y\} - \{x\}$ is a maximum stable set in $G - \{x\}$, i.e., x is not α-critical in G.

(ii) $\Rightarrow$ *(iii)* Now, clearly $S = \{x : x \textit{ is a pendant vertex in } G\} \in \Omega(G)$ and let denote $M = \{xy : x \in S \textit{ and } y \in N(x)\}$. By Proposition 2.2, $G = S * H$ and if some $z \in V(H)$ is not matched by M, then $S \cup \{z\}$ is a stable set larger than S, a contradiction. Hence, we get that $|M| = \mu(G) = |V(H)|$, i.e., G is a König-Egervary graph. According to Theorem 3.5, G is also α^+-stable, because is blossom-free with respect to M and another perfect matching does not exist.

(iii) $\Rightarrow$ *(i)* According to Theorem 3.5, G has a perfect matching M, and since $\{x : x \textit{ is a pendant vertex in } G\} \in \Omega(G)$, M consists of all the pendant edges of G. ■

Lemma 4.8 *If G is a König-Egervary graph, then*

$$N(\cap\{S : S \in \Omega(G)\}) = \cap\{V - S : S \in \Omega(G)\}.$$

Proof. Let denote $A = \cap\{S : S \in \Omega(G)\}$ and $B = \cap\{V - S : S \in \Omega(G)\}$. If $v \in N(A)$, then clearly $v \notin S$, for any $S \in \Omega(G)$, i.e., $N(A) \subseteq B$. Let M be a maximum matching of G and $x \in B$. According to Lemma 2.3, $M \subset (S, V(G) - S)$ holds for any $S \in \Omega(G)$, and by Proposition 2.2, we have also $|M| = |V(G) - S|$. Since $x \in B$, it follows that there is $xy \in M$, and hence, Lemma 2.5 implies that $y \in S$, for any $S \in \Omega(G)$, i.e., $y \in A$. Consequently, we get that $x \in N(A)$, and because x was an arbitrary vertex of B, it results $B \subseteq N(A)$, and this completes the proof. ■

Lemma 4.8 is not true for general graphs; e.g., the graph in Figure 5.

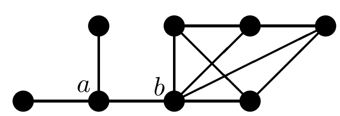

Figure 5: $N(\cap\{S : S \in \Omega(G)\}) = \{a\} \neq \{a, b\} = \cap\{V - S : S \in \Omega(G)$, and G is a non-König-Egervary graph without perfect matchings.

Lemma 4.9 *If G is a König-Egervary graph and M is a maximum matching, then M matches $N(\cap\{S : S \in \Omega(G)\})$ into $\cap\{S : S \in \Omega(G)\}$.*

Proof. In accordance with Proposition 2.2, G can be written as $G = S * H$, where $S \in \Omega(G)$, $H = (V(H), E(H)) = G[V - S]$, and $|V(H)| = \mu(G)$. By Lemma 2.3, $M \subset (S, V(G) - S) = (S, V(H))$, and, clearly, $N(\cap\{S : S \in \Omega(G)) \subset V(G) - S$. Hence, any $x \in N(\cap\{S : S \in \Omega(G))$ is matched with some $y \in S$. Moreover, according to Lemma 2.5, if x belongs to no maximum stable set of G, then $y \in \cap\{S : S \in \Omega(G)\}$. Therefore, M matches $N(\cap\{S : S \in \Omega(G))$ into $\cap\{S : S \in \Omega(G)$. ■

Theorem 4.10 *If G is a König-Egervary graph, then G has a perfect matching if and only if $|\cap\{S : S \in \Omega(G)\}| = |\cap\{V - S : S \in \Omega(G)\}|$.*

Proof. Let M be a perfect matching of G. Then, for any edge $e = xy \in M$, we have that $x \in \cap\{S : S \in \Omega(G)\}$ if and only if $y \in \cap\{V - S : S \in \Omega(G)\}$. Consequently, we get that $|\cap\{S : S \in \Omega(G)\}| = |\cap\{V - S : S \in \Omega(G)\}|$.

Conversely, assume that $|\cap\{S : S \in \Omega(G)\}| = |\cap\{V - S : S \in \Omega(G)\}|$. Let $A = \cap\{S : S \in \Omega(G)\}, B = \cap\{V - S : S \in \Omega(G)\}$, and $G_0 = G - N[A]$. By Proposition 2.2, $G = S * H$, where $H = (V(H), E(H)) = G[V - S]$ has $|V(H)| = \mu(G)$, and $S \in \Omega(G)$. If M is a maximum matching, then by Lemma 4.9, M matches $N(A)$ into A. Hence, $|A| \geq |N(A)|$. Since, by Lemma 4.8, $N(A) = B$, we get that $|A| \geq |N(A)| = |B|$, and consequently $|A| = |N(A)|$. Therefore the restriction M_1 of M on $G[A \cup N(A)]$ is a perfect matching.

For a König-Egervary graph $G, \alpha(G_0) = \alpha(G) - |A|, \mu(G_0) = \mu(G) - |N(A)|$. In our case, when $|A| = |N(A)|$, it means that G_0 is a König-Egervary graph, as well.

Moreover, $|\cap\{S : S \in \Omega(G_0)\}| = 0$, and consequently according to Theorem 1.1 G_0 is an α^+-stable graph. By Theorem 3.2, G_0 has a perfect matching, say M_0, which together with M_1 builds a perfect matching of G. ■

It is interesting to mention that there exist non-König-Egervary graphs enjoying the equality $|\cap\{S : S \in \Omega(G)\}| = |\cap\{V - S : S \in \Omega(G)\}|$ without perfect matchings (e.g., the graph in Figure 5).

Combining Theorem 4.10 and Corollary 4.1 we obtain:

Corollary 4.11 *If G is bipartite, then $|\cap\{S : S \in \Omega(G)\}| = |\cap\{V - S : S \in \Omega(G)\}|$ if and only if $|\cap\{S : S \in \Omega(G)\}| = |\cap\{V - S : S \in \Omega(G)\}| = 0$.*

5 Conclusions and future work

In this paper we return the attention of the reader to the notion of a König-Egervary graph. We state several properties of König-Egervary graphs, showing that these graphs give a fruitful developing of the bipartite graphs theory. Our main findings refer to the α^+-stability of König-Egervary graphs. These results generalize some previously known statements for trees and bipartite graphs. In addition, we characterize those König-Egervary graphs for which "to be blossom-free relative to some perfect matching" is equivalent to "to be blossom-free relative to any perfect matching". This condition is similar both in form and spirit to Sterboul's characterization of König-Egervary graphs. An obvious question arises: which König-Egervary graphs are α^--stable (i.e., have stability number insensitive to deletion of any edge)? It would be also interesting to describe the König-Egervary graphs that are both α^--stable and α^+-stable.

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